

ON HARISH-CHANDRA THEORY OF GLOBAL NONSYMMETRIC FUNCTIONS

IVAN CHEREDNIK [†]

ABSTRACT. This paper is devoted to the Harish-Chandra-type decomposition of the global nonsymmetric spherical functions in terms of their asymptotic expansions and the q, t -generalization of the celebrated c -function. This is for any reduced root systems in the q, t -setting; we pay special attention to the case of A_1 , where this decomposition is very explicit.

Key words: *Hecke algebras; Macdonald polynomials; spherical functions; Harish-Chandra theory; Dunkl operators; hypergeometric functions.*

MSC (2010): 20C08, 33C52, 33C67, 33D52, 33D67

CONTENTS

0. INTRODUCTION	2
1. THE CASE OF A_1	3
2. GENERAL THEORY	10
REFERENCES	19

[†] July 22, 2014.
Simons Foundation.

Partially supported by NSF grant DMS-1363138 and the

0. INTRODUCTION

In this paper, we obtain the Harish-Chandra-type asymptotic decomposition of the *global nonsymmetric spherical* q, t -function $G(X, \Lambda)$ from [Ch4]. Given a Weyl chamber \mathcal{C} , $G(X, \Lambda)$ is represented as a weighted sum of its asymptotic expansions for $\mathcal{C} \ni \Re(x) \sim \infty$ for Λ and its translation by $w \in W$ for the Weyl group W , where we set $X = q^x$. The weight functions are given in terms of $\sigma(\Lambda)$ from [Ch5], a q, t -generalization of the celebrated Harish-Chandra c -function. Only reduced root systems will be considered.

The role of $\sigma(\Lambda)$ here is similar to that in the symmetric case; see [Ch5, St2] for the corresponding Harish-Chandra theory. However the nonsymmetric asymptotic decomposition is not a W -symmetrization in the Λ -space since $G(X, \Lambda)$ is not W -invariant. The asymptotic expansions of $G(X, w(\Lambda))$ are not connected with each other for $w \in W$ in any direct way. They form a W -spinor solving the corresponding difference Dunkl eigenvalue problem in the terminology of [CM, CO1, CO2]. This is parallel to the (differential) nonsymmetric theory in [Op].

Using the (spinor) Dunkl eigenvalue problem is of obvious significance here. It provides the asymptotic expansions and the existence of the required decomposition with undetermined coefficients. The first of these applications can be actually replaced by using the technique of intertwining operators from [Ch3, HHL, RY, OS], the second is very classical. The Dunkl eigenvalue problem (and W -spinors) will be discussed in our further paper(s); we do not need it too much in this particular paper, as well as the related DAHA theory.

We try to make this work short and as focused as possible. Also, to simplify the setup, only the negative Weyl chamber $\mathcal{C}_- \ni \Re(x)$ will be considered, where $\Re(x, \alpha) < 0$ for all positive roots α . Actually it suffices to establish the σ -decomposition only in some open X -set (all involved function are meromorphic). Arbitrary Weyl chambers will be hopefully considered in our further works.

The function $G(X, \Lambda)$ is given by a series that converges anywhere. However its asymptotic expansion in $\mathcal{C}_- \ni \Re(x) \sim \infty$ is only meromorphic and has certain finite radius of convergence (dependent on q, t). Here Λ can be arbitrary avoiding singularities. For sufficiently large negative $\Re(x)$, the cancelation of the Λ -singularities in the σ -decomposition of $G(X, \Lambda)$ is an impressive application of the theory of global functions. We note that generally there are no a priori ways for establishing such a cancelation in the (differential) Heckman-Opdam theory of the hypergeometric function [HO, Op]. Now it can be (potentially) deduced from the difference theory by taking the limit $q \rightarrow 1$.

We essentially follow [Ch5], switching from the asymptotic expansions of the symmetric Macdonald polynomials there to those for the nonsymmetric Macdonald polynomials. The method from [St2] is restricted to the symmetric theory; the difference Cherednik-Matsuo isomorphism theorem is used there, connecting the eigenvalue problem for the Macdonald operators with the difference *AKZ-system* [Ch1, CM, MS]. This approach is not needed in the nonsymmetric theory because the Dunkl operators can be used instead. As a matter of fact, the (spinor) Dunkl eigenvalue problem is the key in the justification of the Cherednik-Matsuo theorems. Note that [St2] includes non-reduced root systems (the present one is restricted to the reduced case).

At almost any levels, the nonsymmetric direction simplifies, clarifies and generalizes the symmetric one. This is especially true in the difference theory, which already significantly changed the classical harmonic analysis on symmetric spaces. This is fully applicable to this paper. However we do not have any geometric interpretation of the nonsymmetric σ -decomposition formula at the moment. Moreover, the geometric meaning of the E -polynomials themselves for generic q, t remains essentially unknown; the symmetrization is generally needed here (apart from several limiting cases).

We pay special attention to A_1 in this paper, where our formula extends that from [CO1]. Here the decomposition is very explicit and the connection with the classical *basic hypergeometric function* can be readily seen. Also, such an explicit formula makes it possible to analyze its behavior at $|q| = 1$. This was touched upon in [CO1] (A_1 , the symmetric case), but will not be discussed in the present paper.

Acknowledgements. The author is thankful to M. Duflo, D. Kazhdan, E. Opdam, J. Stokman for various discussions of the Harish-Chandra theory and M. Finkelberg for a discussion of [BFS]. This paper was stimulated by author's talks at RIMS and in Tokyo University in 2014, as well as B. Feigin's question concerning the existence of W -orbit-sum formulas for the nonsymmetric Macdonald polynomials (see (2.22) at the end). The paper was mainly written at RIMS; the author thanks Hiraku Nakajima and RIMS for the invitation and hospitality and the Simons Foundation (which made this visit possible).

1. THE CASE OF A_1

We will begin this paper with the case of A_1 , when the formulas for the E -polynomials and the related functions are (exceptionally)

explicit. Let us recall the definition of the global function from [Ch5] in the case of A_1 . See Theorem 5.4 there and also [CO1].

The constant term functional, the coefficient of X^0 of a Laurent series f or a polynomial in terms of $X^{\pm 1}$ will be denoted by $\langle \cdot \rangle$. The μ -function, the measure that makes the E -polynomials orthogonal, is the following truncated theta function:

$$(1.1) \quad \mu(X; q, t) \stackrel{\text{def}}{=} \prod_{j=0}^{\infty} \frac{(1 - q^j X^2)(1 - q^{j+1} X^{-2})}{(1 - tq^j X^2)(1 - tq^{j+1} X^{-2})}.$$

We will mainly need its renormalization

$$(1.2) \quad \mu_{\circ} \stackrel{\text{def}}{=} \mu / \langle \mu \rangle = 1 + \frac{t-1}{1-qt} (X^2 + qX^{-2}) + \dots,$$

$$\text{where } \langle \mu \rangle = \prod_{j=1}^{\infty} \frac{(1 - tq^j)^2}{(1 - t^2 q^j)(1 - q^j)}.$$

The series μ_{\circ} is $*$ -invariant for the conjugation

$$X^* = X^{-1}, \quad (q^{1/2})^* = q^{-1/2}, \quad (t^{1/2})^* = t^{-1/2}.$$

The Demazure-Lusztig operator T and the difference Dunkl operator Y are as follows:

$$T = t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{X^2 - 1} \cdot (s - 1), \quad Y = s \Gamma T,$$

$$s(X^n) = X^{-n}, \quad \Gamma(X^n) = q^{n/2} X^n \quad \text{for } n \in \mathbb{Z}.$$

They naturally act in the polynomial DAHA representation, which is $\mathcal{V} \stackrel{\text{def}}{=} \mathbb{Z}[q^{\pm 1/2}, t^{\pm 1/2}][X^{\pm 1}]$. We will sometimes set $X = q^x$; then $s(x) = -x$, $\Gamma(f(x)) = f(x + 1/2)$.

The *nonsymmetric Macdonald polynomials*, also commonly called *E-polynomials*, are uniquely determined from the eigenvalue problem

$$(1.3) \quad Y(E_n) = q^{-n_{\sharp}} E_n \quad \text{for } n \in \mathbb{Z}, \quad t \stackrel{\text{def}}{=} q^k,$$

$$(1.4) \quad n_{\sharp} = \begin{cases} \frac{n+k}{2} & \text{for } n > 0, \\ \frac{n-k}{2} & \text{for } n \leq 0 \end{cases}, \quad \text{note that } 0_{\sharp} = -\frac{k}{2},$$

where the normalization is $E_n = X^n + \text{“lower terms”}$. By “lower terms”, we mean polynomials in terms of $X^{\pm m}$ as $|m| < n$ and, additionally, $X^{|n|}$ for $n < 0$. It gives a filtration in \mathcal{V} preserved by Y . These polynomials (for any reduced root systems) are due to Opdam in the differential setting (he mentions a contribution of Heckman) and Macdonald for integral $k \geq 0$; see [Op, Ma, Ch2, Ch3].

Obviously, $E_0 = 1$, $E_1 = X$. Let us provide the formulas for the E -polynomials ($n > 0$):

$$(1.5) \quad \begin{aligned} E_{-n} = & X^{-n} + X^n \frac{1-t}{1-tq^n} + \sum_{j=1}^{[n/2]} X^{2j-n} \prod_{i=0}^{j-1} \frac{(1-q^{n-i})}{(1-q^{1+i})} \frac{(1-tq^i)}{(1-tq^{n-i})} \\ & + \sum_{j=1}^{[(n-1)/2]} X^{n-2j} \frac{(1-tq^j)}{(1-tq^{n-j})} \prod_{i=0}^{j-1} \frac{(1-q^{n-i})}{(1-q^{1+i})} \frac{(1-tq^i)}{(1-tq^{n-i})}, \end{aligned}$$

$$(1.6) \quad \begin{aligned} E_n = & X^n + \sum_{j=1}^{[n/2]} X^{2j-n} q^{n-j} \frac{(1-q^j)}{(1-q^{n-j})} \prod_{i=0}^{j-1} \frac{(1-q^{n-i-1})}{(1-q^{1+i})} \frac{(1-tq^i)}{(1-tq^{n-i-1})} \\ & + \sum_{j=1}^{[(n-1)/2]} X^{n-2j} q^j \prod_{i=0}^{j-1} \frac{(1-q^{n-i-1})}{(1-q^{1+i})} \frac{(1-tq^i)}{(1-tq^{n-i-1})}. \end{aligned}$$

These are formulas (3.10 - 3.12) from [CM] and [CO1]. One can present them as infinite series, which are actually finite and will terminate automatically. This is important for what will follow. We note that T. Koornwinder was the first to explicitly calculate the formulas for the E -polynomials for A_1 . One has:

$$(1.7) \quad E_{-n} = \sum_{j=0}^{\infty} X^{n-2j} \frac{(1-tq^j)}{(1-tq^{n-j})} \prod_{i=0}^{j-1} \frac{(1-q^{n-i})}{(1-q^{1+i})} \frac{(1-tq^i)}{(1-tq^{n-i})},$$

$$(1.8) \quad E_n = \sum_{j=0}^{\infty} X^{n-2j} q^j \prod_{i=0}^{j-1} \frac{(1-q^{n-i-1})}{(1-q^{1+i})} \frac{(1-tq^i)}{(1-tq^{n-i-1})} \quad (n > 0),$$

where the actual summation will be till $j=n$ in the first formula (which works for $n=0$) and till $j=n-1$ in the second; the products there will vanish otherwise.

Setting $\theta(X) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} q^{n^2/4} X^n$, the *global nonsymmetric function* for A_1 is defined in [Ch4] as follows:

$$(1.9) \quad \begin{aligned} \frac{\theta(X)\theta(\Lambda)}{\theta(t^{1/2})} G(X; \Lambda) &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{4}} t^{\frac{|n|}{2}} \frac{E_n^*(X) E_n(\Lambda)}{\langle E_n E_n^* \mu_{\circ} \rangle} \\ &= \Psi(X, \Lambda) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{4}} t^{\frac{|n|}{2}} \frac{E_n(X) E_n^*(\Lambda)}{\langle E_n E_n^* \mu_{\circ} \rangle}. \end{aligned}$$

The following properties of the G -function are from [Ch4] (for any reduced root systems):

$$(1.10) \quad Y(G) = \Lambda^{-1} G, \quad X^{-1} G = Y_\Lambda(G), \quad T(G) = T_\Lambda(G),$$

$$(1.11) \quad G(X, \Lambda = q^{n\sharp}) = \prod_{i=1}^{\infty} \frac{1 - tq^i}{1 - q^i} \frac{E_n(X)}{E_n(t^{-1/2})}, \quad n \in \mathbb{Z},$$

$$(1.12) \quad E_n(t^{-\frac{1}{2}}) = t^{-\frac{|n|}{2}} \prod_{0 < j < |\tilde{n}|} \frac{1 - q^j t^2}{1 - q^j t}, \quad |\tilde{n}| = \begin{cases} |n| + 1 & \text{for } n \leq 0, \\ n & \text{for } n > 0 \end{cases}.$$

Here Y_Λ and T_Λ are Y, T where X is replaced by Λ . The relations from (1.10), the Shintani-type formula (1.11) and the $X \leftrightarrow \Lambda$ symmetry in (1.9) are the key ingredients of our approach, extending that in [Ch5].

The symmetric Macdonald polynomials $P_n(n \geq 0)$ for A_1 , which are the Rogers polynomials, are given by the formulas $P_n = E_{-n} + \frac{t - tq^n}{1 - tq^n} E_n$. Accordingly,

$$(1.13) \quad \begin{aligned} (1+t) \frac{P_n(X)}{P_n(t^{1/2})} &= \frac{1 - t^2 q^n}{1 - tq^n} \frac{E_{-n}(X)}{E_{-n}(t^{-1/2})} + \frac{t - tq^n}{1 - tq^n} \frac{E_n(X)}{E_n(t^{-1/2})} \quad \text{for } n \geq 0, \\ (1+t)F(X, \Lambda) &= \frac{t - \Lambda^{-2}}{1 - \Lambda^{-2}} G(X, \Lambda^{-1}) + \frac{t - \Lambda^2}{1 - \Lambda^2} G(X, \Lambda) \quad \text{for} \\ \frac{\theta(X)\theta(\Lambda)}{\theta(t^{1/2})} F(X, \Lambda) &= \Phi(X, \Lambda) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} q^{\frac{n^2}{4}} t^{\frac{n}{2}} \frac{P_n(X)P_n(\Lambda)}{\langle P_n P_n \mu_o \rangle} \end{aligned}$$

apart from the zeros of $\theta(X)\theta(\Lambda)$. See the last three formulas of [Ch4].

Theorem 1.1. *For the function $G(X, \Lambda)$ from (1.9), let us assume that $|q| < 1$ and $|X| > |t|^{-1/2}|q|^{1/2}$. For $\langle \mu \rangle$ from (1.2),*

$$(1.14) \quad \begin{aligned} \Psi(X, \Lambda) &= \langle \mu \rangle \left(\sigma\left(\frac{1}{\Lambda}\right) \theta(X\Lambda t^{\frac{1}{2}}) \Xi_{-}(X, \Lambda) + \sigma(\Lambda) \theta\left(\frac{X}{\Lambda} t^{\frac{1}{2}}\right) \Xi_{+}(X, \frac{1}{\Lambda}) \right), \\ \Xi_{-}(X, \Lambda) &= \frac{1-t}{1-t\Lambda^{-2}} + \sum_{j=1}^{\infty} \frac{1-tq^j}{1-t\Lambda^{-2}} \left(\frac{q}{t}\right)^j X^{-2j} \prod_{s=1}^j \frac{(1-tq^{s-1})(1-tq^{s-1}\Lambda^2)}{(1-q^s)(1-q^s\Lambda^2)}, \\ \Xi_{+}(X, \Lambda) &= 1 + \sum_{j=1}^{\infty} \frac{1-tq^j\Lambda^2}{1-t\Lambda^2} \left(\frac{q}{t}\right)^j X^{-2j} \prod_{s=1}^j \frac{(1-tq^{s-1})(1-tq^{s-1}\Lambda^2)}{(1-q^s)(1-q^s\Lambda^2)}, \\ \text{where } \sigma(\Lambda) &= \prod_{j=0}^{\infty} \frac{1-tq^j\Lambda^2}{1-q^j\Lambda^2} = 1 + \sum_{j=1}^{\infty} \Lambda^{2j} \prod_{s=1}^j \frac{1-tq^{s-1}}{1-q^s}, \end{aligned}$$

which is a q, t -generalization of the Harish-Chandra c -function. Using (1.13), we arrive at Theorem 2.3 from [CO1] for $X \mapsto X^{-1}$:

$$\begin{aligned} \Phi(X, \Lambda) = & \langle \mu \rangle \sigma(\frac{1}{\Lambda}) \frac{\theta(X \Lambda t^{\frac{1}{2}})}{1+t} \sum_{j=0}^{\infty} \left(\frac{q}{t}\right)^j X^{-2j} \prod_{s=1}^j \frac{(1-tq^{s-1})(1-tq^{s-1}\Lambda^2)}{(1-q^s)(1-q^s\Lambda^2)} \\ & + \langle \mu \rangle \sigma(\Lambda) \frac{\theta(X \Lambda^{-1} t^{1/2})}{1+t} \sum_{j=0}^{\infty} \left(\frac{q}{t}\right)^j X^{-2j} \prod_{s=1}^j \frac{(1-tq^{s-1})(1-tq^{s-1}\Lambda^{-2})}{(1-q^s)(1-q^s\Lambda^{-2})}. \end{aligned}$$

Proof. Let us present $X^{-n}E_{\pm n}$ in terms of Λ . Setting

$$\begin{aligned} (1.15) \quad \tilde{\Xi}_+(X, \Lambda) & \stackrel{\text{def}}{=} \Xi_+(X, \Lambda), \quad \tilde{\Xi}_-(X, \Lambda) \stackrel{\text{def}}{=} \Xi_-(X, \Lambda) \frac{1-t\Lambda^{-2}}{1-\Lambda^{-2}} \\ & = \frac{1-t}{1-\Lambda^{-2}} + \sum_{j=1}^{\infty} \frac{1-tq^j}{1-\Lambda^{-2}} \left(\frac{q}{t}\right)^j X^{-2j} \prod_{s=1}^j \frac{(1-tq^{s-1})(1-tq^{s-1}\Lambda^2)}{(1-q^s)(1-q^s\Lambda^2)}, \end{aligned}$$

we claim that

$$(1.16) \quad \frac{E_{n+1}}{X^{n+1}} = \tilde{\Xi}_+(X, \Lambda^2 = \frac{1}{tq^{n+1}}), \quad \frac{E_{-n}}{X^n} = \tilde{\Xi}_-(X, \Lambda^2 = \frac{1}{tq^n}) \quad \text{for } n \geq 0.$$

These two formulas are direct from (1.7, 1.8), as well as

$$\begin{aligned} (1.17) \quad X^n E_{-n} & = \hat{\Xi}_+(X, \Lambda^2 = \frac{1}{tq^n}) \quad \text{for } n \geq 0, \quad \text{where} \\ \hat{\Xi}_+(X, \Lambda) & \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \frac{X^{2j}}{t^j} \prod_{s=1}^j \frac{(1-tq^{s-1})(1-tq^{s-1}\Lambda^2)}{(1-q^s)(1-q^s\Lambda^2)}. \end{aligned}$$

Following [Ch4] and using $Y(G) = \Lambda^{-1}G$ from (1.10) (instead of the Macdonald eigenvalue problem there), we obtain that

$$(1.18) \quad \frac{\Psi(X, \Lambda)}{\langle \mu \rangle} = \alpha\left(\frac{1}{\Lambda}\right) \theta(X \Lambda t^{\frac{1}{2}}) \Xi_-(X, \Lambda) + \beta(\Lambda) \theta\left(\frac{X}{\Lambda} t^{\frac{1}{2}}\right) \Xi_+(X, \frac{1}{\Lambda})$$

for certain functions $\alpha(\Lambda), \beta(\Lambda)$, which do not depend on X (see the general case below). They can be determined as follows.

Let us find $\alpha(\Lambda)$ and $\beta(\Lambda)$. We will use the relation $\theta(Xq^{m/2}) = X^{-m}q^{-m^2/4}\theta(X)$ and the $\frac{1}{2}\mathbb{Z}$ -periodicity of $q^{x^2}\theta(X)$ in terms of x defined from $X = q^x$; the latter periodicity readily implies the former relation. We set $\Lambda_n \stackrel{\text{def}}{=} t^{\frac{1}{2}}q^{\frac{n}{2}}$ for $n > 0$ and $\Lambda_{-n} \stackrel{\text{def}}{=} (t^{\frac{1}{2}}q^{\frac{n}{2}})^{-1}$ for $n \geq 0$.

Applying the Shintani-type formula (1.11) for $E_{-n}(n \in \mathbb{Z}_+)$, let us establish the theorem when $\Lambda = \Lambda_{-n}$. One has:

$$\begin{aligned}
 (1.19) \quad & \prod_{i=1}^n \frac{1-t^2 q^i}{1-q^i t} \prod_{i=1}^{\infty} \frac{1-q^i}{1-tq^i} t^{-\frac{n}{2}} G(X, \Lambda_{-n} = (tq^n)^{-\frac{1}{2}}) = E_{-n}(X) \Rightarrow \\
 & \prod_{i=1}^n \frac{1-t^2 q^i}{1-q^i t} \prod_{i=1}^{\infty} \frac{1-q^i}{1-tq^i} \frac{\theta(X)\theta(t^{-\frac{1}{2}}q^{-\frac{n}{2}})}{\theta(t^{1/2})} G(X, (tq^n)^{-\frac{1}{2}}) = \theta(X\Lambda_{-n}t^{\frac{1}{2}}) \frac{E_{-n}}{X^n}, \\
 & \prod_{i=1}^n \frac{1-t^2 q^i}{1-q^i t} \prod_{i=1}^{\infty} \frac{1-q^i}{1-tq^i} \Psi(X, (tq^n)^{-\frac{1}{2}}) = \theta(X\Lambda_{-n}t^{\frac{1}{2}}) \frac{E_{-n}(X)}{X^n}.
 \end{aligned}$$

The last equality is exactly the decomposition for $\Psi(X, \Lambda = \Lambda_n)$ from (1.14). Indeed, using the product formula for $\langle \mu \rangle$ from (1.2) and upon the division of the last equality by $\sigma(\Lambda_n)\theta(X\Lambda_{-n}t^{\frac{1}{2}})$,

$$\begin{aligned}
 \frac{\tilde{\Xi}_{-}(X, \Lambda_{-n})}{\sigma(\Lambda_n)} &= \langle \mu \rangle \prod_{i=1}^n \frac{1-t^2 q^i}{1-q^i t} \prod_{i=1}^{\infty} \frac{1-q^i}{1-tq^i} \Xi_{-}(X, \Lambda_{-n}) \\
 &= \prod_{i=1}^{\infty} \frac{(1-tq^i)^2}{(1-t^2 q^i)(1-q^i)} \prod_{i=1}^n \frac{1-t^2 q^i}{1-tq^i} \prod_{i=1}^{\infty} \frac{1-q^i}{1-tq^i} \Xi_{-}(X, \Lambda_{-n}) \\
 &= \prod_{i=n+1}^{\infty} \frac{1-tq^i}{1-t^2 q^i} \Xi_{-}(X, \Lambda_{-n}) = \frac{1-t^2 q^n}{1-tq^n} \frac{\Xi_{-}(X, \Lambda_{-n})}{\sigma(\Lambda_n)},
 \end{aligned}$$

which is the relation between Ξ and $\tilde{\Xi}$ from (1.15).

Thus $\alpha(\Lambda_n) = \sigma(\Lambda_n)$ and $\beta(\Lambda_{-n}) = 0$ for $n \in \mathbb{Z}_+$. Similarly, one checks that $\alpha(\Lambda_{-n}) = 0$ and $\beta(\Lambda_n) = \sigma(\Lambda_n)$ for $n > 0$. This is actually sufficient to determine the coefficients α, β uniquely by analyticity considerations, which can be also established as follows.

Classical method. Let assume that $|X|$ approaches ∞ in (1.18) provided $|\Lambda| > 1$ and that the term with β does not contribute to the resulting *asymptotic* expansion. This is the classical track; we set $X = q^x, \Lambda = q^\lambda$ providing $\Re(x), \Re(\lambda) < 0$ and $\Re(x\lambda) > 0$. Under these assumptions, let $\Psi_{-}(X, \Lambda) \stackrel{\text{def}}{=} \langle \mu \rangle^{-1} \Psi(X, \Lambda) \theta(X\Lambda t^{\frac{1}{2}})^{-1}$. Then $\Psi_{-}(X, \Lambda)$ approaches $\alpha(\Lambda^{-1})(1-t)$ as $|X| \rightarrow \infty$. On the other hand, this limit can be readily calculated using the symmetry $X \leftrightarrow \Lambda$. Namely,

$$\begin{aligned}
 \lim_{|X| \rightarrow \infty} \Psi_{-}(X, \Lambda) &= \lim_{n \rightarrow \infty} \Psi_{-}(\Lambda, X_{-n} = t^{-1/2} q^{-n/2}) = \tilde{\Xi}_{-}(\Lambda, |X| \rightarrow \infty) \\
 (1.20) \quad &= 1-t + \sum_{j=1}^{\infty} (1-tq^j) \Lambda^{-2j} \prod_{s=1}^j \frac{1-tq^{s-1}}{1-q^s} = (1-t)\sigma(\Lambda^{-1}).
 \end{aligned}$$

Use the formula for $\tilde{\Xi}(X, \Lambda)$ and the expansion of σ from (1.14).

Similarly, sending $|X| \rightarrow \infty$ in (1.18) for $|\Lambda| < 1$ provides the formula for the β -coefficient. Now $\Re(\lambda) < 0 > \Re(x)$ and we the relation $\Re(x\lambda) < 0$ is imposed. We set

$$(1.21) \quad \Psi_{-}^{+}(X, \Lambda) \stackrel{\text{def}}{=} \frac{\Psi(X, \Lambda)}{\langle \mu \rangle} \theta\left(\frac{\Lambda}{X} t^{\frac{1}{2}}\right)^{-1}, \quad \Psi_{+}^{-} \stackrel{\text{def}}{=} \frac{\Psi(X, \Lambda)}{\langle \mu \rangle} \theta\left(\frac{X}{\Lambda} t^{\frac{1}{2}}\right)^{-1},$$

which are proportional to the expansions of $X^n E_{-n}$ and $X^{-n} E_n$ as $\Lambda = (tq^n)^{\pm 1/2}$ in terms of X^m for $m \in \pm \mathbb{Z}_+$ (with explicit coefficients of proportionality due to the Shintani-type formulas). Parallel to the above consideration of Ψ_{-} ,

$$(1.22) \quad \lim_{|X| \rightarrow \infty} \Psi_{+}^{-}(X, \Lambda) = \beta(\Lambda) \Xi_{+}(X \rightarrow \infty, \frac{1}{\Lambda}) = \beta(\Lambda).$$

Then we employ the $X \leftrightarrow \Lambda$ symmetry and use (1.17):

$$\begin{aligned} \beta(\Lambda) &= \lim_{|X| \rightarrow \infty} \Psi_{-}^{+}(\Lambda, X) = \lim_{n \rightarrow \infty} \Psi_{-}^{+}(\Lambda, X = X_{-n}) = \hat{\Xi}_{-}(\Lambda, X \rightarrow \infty) \\ &= \lim_{|X| \rightarrow \infty} \sum_{j=0}^{\infty} \frac{\Lambda^{2j}}{t^j} \prod_{s=1}^j \frac{(1-tq^{s-1})(1-tq^{s-1}X^2)}{(1-q^s)(1-q^sX^2)} = \sigma(\Lambda). \end{aligned}$$

□

We note that $\langle \mu \rangle$ appears in this theorem as a result of direct calculation based on the constant term and the evaluation identities for the nonsymmetric Macdonald polynomials and the Shintani-type formula for global functions. The presence of $\langle \mu \rangle$ can be seen in a more conceptual way, but we will not discuss this in the present paper. Note that the $X \leftrightarrow \Lambda$ -symmetry of $G(X, \Lambda)$, which is the key, has no counterpart in the differential Harish-Chandra theory (though holds in the rational limit, the theory of nonsymmetric Hankel transform).

The approach we use can be generalized to arbitrary root systems. Following [Ch5], one can calculate the expansions of $X^{u(b)} E_{w(b)}$ for the nonsymmetric Macdonald polynomials for any $b \in P_{-}$ and $u, w \in W$ (see [Ch1] and below). They will be in terms of X_{α} for the roots $\alpha \in u(R_{+})$ and will coincide with $\sigma(X)$ up to certain explicit factors. The theorem above was stated only for $|X| > 1$. Following (1.21), the functions Ξ_{-}, Ξ_{+} can be naturally denoted by Ξ_{-}^{-} and Ξ_{+}^{-} . Then the expansions of Ψ for $|X| < 1$ will be in terms of Ξ_{\pm}^{+} . The calculation of these expansions is very similar, but we will omit this in the present paper. Let us go to the general theory.

2. GENERAL THEORY

The extension of Theorem 1.1 to arbitrary reduced root systems (and its proof) is relatively straightforward. We follow [Ch2] (concerning E -polynomials and their main properties), [Ch4] (for the global functions), [Ch5] (the σ -function) and [Ch1]. See [St1] for the global functions in the $C^\vee C_n$ -case.

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type A, B, \dots, G_2 with respect to a euclidean form (z, z') on $\mathbb{R}^n \ni z, z'$, W the Weyl group generated by the reflections s_α , R_+ the set of positive roots corresponding to fixed simple roots $\alpha_1, \dots, \alpha_n$; $R_- = -R_+$. The form is normalized by the condition $(\alpha, \alpha) = 2$ for *short* roots. The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the coroots $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Replacing \mathbb{Z} by $\mathbb{Z}_+ = \{m \in \mathbb{Z}, m \geq 0\}$, we obtain Q_+, P_+ ; we will constantly use $P_- = -P_+$. Let $\iota(b) = b^\iota = -w_0(b)$ for the element w_0 of the maximal length in W .

Setting $\nu_\alpha \stackrel{\text{def}}{=} (\alpha, \alpha)/2$, the vectors $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the *twisted affine root system* $\tilde{R} \supset R$ ($z \in \mathbb{R}^n$ are identified with $[z, 0]$). The corresponding set \tilde{R}_+ of positive roots is $R_+ \cup \{[\alpha, \nu_\alpha j], \alpha \in R, j > 0\}$.

The *extended Weyl group* \widehat{W} is $W \ltimes P$, where the corresponding action in \mathbb{R}^{n+1} is

$$(wb)([z, \zeta]) = [w(z), \zeta - (z, b)] \quad \text{for } w \in W, b \in P.$$

The *length* in \widehat{W} is defined as follows:

$$l(\widehat{w}) = |\lambda(\widehat{w})| \quad \text{for } \lambda(\widehat{w}) \stackrel{\text{def}}{=} \tilde{R}_+ \cap \widehat{w}^{-1}(-\tilde{R}_+).$$

For $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$, $0 \leq i \leq n$, we set

$$t_{\tilde{\alpha}} = t_\alpha = t_{\nu_\alpha} = q_\alpha^{k_\nu}, \quad q_{\tilde{\alpha}} = q^{\nu_\alpha}, \quad t_i = t_{\alpha_i}, \quad q_i = q_{\alpha_i},$$

$$\rho_k \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\alpha > 0} k_\alpha \alpha = k_{\text{sht}} \rho_{\text{sht}} + k_{\text{lng}} \rho_{\text{lng}}, \quad \rho_\nu = \frac{1}{2} \sum_{\nu_\alpha = \nu} \alpha = \sum_{\nu_i = \nu, i > 0} \omega_i,$$

where *sht*, *lng* are used instead of ν .

Let \mathcal{V} by the space of Laurent polynomials in terms of $X_b (b \in P)$ satisfying the multiplicative property $X_{b+c} = X_b X_c$; the coefficients are taken from $\mathbb{Z}[q^{\pm 1/m}, t_\nu^{\pm 1/2}]$, where $(P, P) \in \frac{1}{m}\mathbb{Z}$. For any $b \in P$ and

generic q, t , the *nonsymmetric Macdonald polynomials* are uniquely determined by the relations

$$(2.1) \quad Y_a(E_b) = q^{-(a, b - u_b(\rho_k))} E_b \quad \text{and the coefficient of } X_b \in E_b \text{ is } 1$$

for $u_b \in W$ of minimal possible length such that $u_b(b) \in P_-$,

where $Y_a(a \in P)$ are the *difference Dunkl operators*; they act in \mathcal{V} and are pairwise commutative. See Theorem 4.1 from [Ch2] and Proposition 3.3.1 from [Ch1]. The coefficients of E_b belong to $\mathbb{Q}(q, t_\nu)$. Setting $b_\# \stackrel{\text{def}}{=} b - u_b(\rho_k)$, we can rewrite (2.1) as

$$(2.2) \quad Y_a(E_b) = X_b^{-1}(q^{b_\#})E_b, \quad \text{where } X_a(q^b) \stackrel{\text{def}}{=} q^{(a,b)} \quad \text{for } a, b \in P.$$

Let $\xi(\alpha) \stackrel{\text{def}}{=} 0, 1$ respectively for $\alpha \in R_+, R_-$. The Main Theorem in [Ch2] and formula (3.3.16) in [Ch1] state that

$$(2.3) \quad E_b(q^{-\rho_k}) = q^{(\rho_k, b_-)} \prod_{\alpha > 0} \prod_{j=1}^{j(b, \alpha)} \left(\frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right) \quad \text{for } b \in P,$$

$$\text{where } b_- \stackrel{\text{def}}{=} u_b(b) \in P_-, \quad j(b, \alpha) = -(\alpha^\vee, b_-) - \xi(u_b^{-1}(\alpha)).$$

We will also need

$$(2.4) \quad \mu(X; q, t) \stackrel{\text{def}}{=} \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{(1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1})}{(1 - X_\alpha t_\alpha q_\alpha^j)(1 - X_\alpha^{-1} t_\alpha q_\alpha^{j+1})},$$

$$(2.5) \quad \langle \mu \rangle = \prod_{\alpha \in R_+} \prod_{i=1}^{\infty} \frac{(1 - q^{(\rho_k, \alpha) + i\nu_\alpha})^2}{(1 - t_\alpha q^{(\rho_k, \alpha) + i\nu_\alpha})(1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + i\nu_\alpha})},$$

where $\langle \cdot \rangle$ is the constant term functional (the coefficient of X^0). See e.g. (3.3.1), (3.3.2) in [Ch1]. We will use below the norms $\langle E_b E_b^* \mu_\circ \rangle$ for $\mu_\circ \stackrel{\text{def}}{=} \mu / \langle \mu \rangle$ from (3.4.2) there, but do not need the exact formulas for them. The conjugation \star is as follows: $X_b^* = X_b^{-1}$, $q^* = q^{-1}$, $t_\nu^* = t_\nu^{-1}$.

We will omit the definition of the symmetric Macdonald polynomials $P_b(b \in P_-)$ in this paper; see (3.3.12) and (3.3.14) from [Ch1] and (3.4.3) there for the norms $\langle P_b P_{b^\vee} \mu_\circ \rangle$. Recall that $b^\vee = -w_0(b)$. Formula (3.3.23) from [Ch1] states:

$$P_{b_-}(q^{-\rho_k}) = P_{b_-}(q^{\rho_k}) = E_{b_+}(q^{-\rho_k}) \prod_{\alpha > 0} \left(\frac{1 - t_\alpha X_\alpha(q^{\rho_k})}{1 - X_\alpha(q^{\rho_k})} \right).$$

The following formula is from (4.13) [Ch2] and (3.3.15) from [Ch1]:

$$(2.6) \quad P_{b_-} = \sum_{c \in W(b_+)} \prod_{(\alpha, c) > 0} \frac{t_\alpha - X_\alpha(q^{c_\#})}{1 - X_\alpha(q^{c_\#})} E_c, \quad \alpha \in R_+.$$

See [Ma] for $k_{\text{sh}} = k_{\text{lng}} \in \mathbb{Z}_+$. We need its variant in terms of

$$E'_b \stackrel{\text{def}}{=} E_b/E_b(q^{-\rho_k}) \text{ and } P''_{b_-} \stackrel{\text{def}}{=} \mathcal{P}_R(t) P_{b_-}/P_{b_-}(q^{-\rho_k}),$$

where $\mathcal{P}_R(t) = \prod_{\alpha > 0} \frac{1 - t_\alpha q^{(\alpha, \rho_k)}}{1 - q^{(\alpha, \rho_k)}}$ is the Poincaré polynomial.

For $\alpha \in R_+$,

$$(2.7) \quad P''_{b_-} = \sum_{c \in W(b_+)} \prod_{u_c^{-1}(\alpha) > 0} \left(\frac{1 - t_\alpha X_\alpha(q^{\rho_k - b_-})}{1 - X_\alpha(q^{\rho_k - b_-})} \right) \prod_{(\alpha, c) > 0} \frac{t_\alpha - X_\alpha(q^{c_\#})}{1 - X_\alpha(q^{c_\#})} E'_c.$$

Switching in the first product to $\beta = u_c^{-1}(\alpha)$, the condition $u_c^{-1}(\alpha) > 0$ for $\alpha > 0$ becomes $(\beta, c) \leq 0$ for $\beta > 0$. Also $\rho_k - b_- = -u_c(c_\#)$. Therefore for $\alpha, \beta > 0$,

$$(2.8) \quad \begin{aligned} P''_{b_-} &= \sum_{c \in W(b_+)} \prod_{(\beta, c) \leq 0} \left(\frac{1 - t_\beta X_\beta^{-1}(q^{c_\#})}{1 - X_\beta^{-1}(q^{c_\#})} \right) \prod_{(\alpha, c) > 0} \frac{t_\alpha - X_\alpha(q^{c_\#})}{1 - X_\alpha(q^{c_\#})} E'_c \\ &= \sum_{c \in W(b_+)} \prod_{\alpha > 0} \frac{t_\alpha - X_\alpha(q^{c_\#})}{1 - X_\alpha(q^{c_\#})} E'_c. \end{aligned}$$

The following construction is from Theorem 5.4 and Corollary 7.3 of [Ch4]. The function $G(X, \Lambda)$ introduced in (2.10) below is called *global nonsymmetric q, t -spherical function*. We will use another set of (pairwise commutative) variables Λ_b . Sometimes it will be convenient to put $X = q^x$. Then $X_b = q^{(x, b)}$ for $b \in P$. Accordingly, $w(X_b) = X_{w(b)}$ for $w \in W$ and $w(f(X)) = f(w^{-1}(X)) = f(q^{w^{-1}(x)})$ for any function f of X (notice w^{-1}). The same notations will be used for $\Lambda = q^\lambda$; we put w_Λ instead of w if this action can be confused with that in terms of X .

We set $\theta(X) \stackrel{\text{def}}{=} \sum_{b \in P} q^{-(b, b)/2} X_b$; obviously $w(\theta(X)) = \theta(X) = th(X^{-1})$ for $w \in W$ and $q^{\frac{(x, x)}{2}} \theta(X)$ is P -periodic in terms of x . If $|q| < 1$, then $\theta(X)$ is convergent and holomorphic anywhere.

Theorem 2.1. (i) *The Laurent series*

$$(2.9) \quad \Psi(X, \Lambda) = \Psi(X, \Lambda; q, t) \stackrel{\text{def}}{=} \sum_{b \in B} q^{(b, b)/2 - (b_-, \rho_k)} \frac{E_b^*(X) E_b(\Lambda)}{\langle E_b E_b^* \mu_o \rangle}$$

is well defined with coefficients in $\mathbb{Q}[t][[q^{\frac{1}{2m}}]]$. For $|q| < 1$, Ψ converges to an entire function of X, Λ , provided t_ν are chosen so that all E -polynomials exist (the conditions $|t_\nu| < 1$ are sufficient). Accordingly,

$$(2.10) \quad G(X, \Lambda) \stackrel{\text{def}}{=} \frac{\theta(q^{\rho_k})}{\theta(X)\theta(\Lambda)} \Psi(X, \Lambda; q, t)$$

is a meromorphic function of X, Λ , which is analytic apart from the zeros of $\theta(X)\theta(\Lambda)$.

(ii) The function $G(X, \Lambda)$ satisfies the relations

$$(2.11) \quad G(X, \Lambda) = G(\Lambda, X), \quad Y_a(G(X, \Lambda)) = \Lambda_a^{-1} G(X, \Lambda) \quad \text{for } a \in P.$$

For an arbitrary $b \in P$, one has the Shintani-type formulas:

$$(2.12) \quad G(X, q^{b^*}) = \frac{E_b(X)}{E_b(q^{-\rho_k})} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q^{(\rho_k, \alpha) + \nu_\alpha j}}{1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha j}} \right).$$

(iii) Let us define the symmetric global function $F(X, \Lambda)$ via

$$(2.13) \quad \frac{\theta(X)\theta(\Lambda)}{\theta(q^{\rho_k})} F(X, \Lambda) = \Phi(X, \Lambda) \stackrel{\text{def}}{=} \sum_{b \in P_-} q^{\frac{(b, b)}{2} - \frac{(b, \rho_k)}{2}} \frac{P_b(X) P_{b^*}(\Lambda)}{\langle P_b P_{b^*} \mu_o \rangle}.$$

Then (2.12) holds with the same coefficient of proportionality:

$$(2.14) \quad F(X, q^{b - \rho_k}) = \frac{P_b(X)}{P_b(q^{\rho_k})} \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left(\frac{1 - q^{(\rho_k, \alpha) + \nu_\alpha j}}{1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha j}} \right) \quad \text{for } b = b_-,$$

$$\text{and (2.8) implies } \mathcal{P}_R(t) F(X, \Lambda) = \sum_{w \in W} \prod_{\alpha > 0} \frac{t_\alpha - \Lambda_{w(\alpha)}}{1 - \Lambda_{w(\alpha)}} G(X, w(\Lambda)).$$

□

The proof is based on the fundamental fact that $G(X, \Lambda)$ represents the Fourier transform of DAHA. For instance, the Shintani-type formula (2.12) follows from formula (3.4.13) in Theorem 3.4.2 [Ch1], which states that the Fourier-images of $E_b/E_b(q^{-\rho_k})$ (with respect to the DAHA-automorphism ε) are the corresponding delta-functions.

Theorem 2.2. (i) Λ -Stabilization. For any given $w \in W$, there exists a unique series $\Xi^{(w)}(X, \Lambda) = \sum_{a \in Q_+} A_a^w(\Lambda) X_a^{-1}$ with the coefficients that are rational functions in terms of q, t_ν, Λ_α such that

$$(2.15) \quad X_{b^*_-} E_b = \Xi^{(w)}(X, \Lambda = q^{b - \rho_k}) \quad \text{for } w = u_b^{-1}, b \in P; \text{ see (2.1).}$$

The functions $A_a^w(\Lambda)$ are regular if $\Lambda_\alpha \notin q_\alpha^\mathbb{Z}$ for every $\alpha \in R_+$ and have formal expansions in the ring $\mathbb{Q}[t_\nu^{\pm 1}][[q_\alpha \Lambda_\alpha, \Lambda_\alpha^{-1}(\alpha > 0)]]$. Assuming $|q| < 1$, let $X = q^x$. There exists a constant $C > 0$ (dependent on q, t_ν) such that the series $\Xi^{(w)}$ converges for any $w \in W$ when $(\Re(x), \alpha) < -C$ for every $\alpha \in R_+$ and Λ can be arbitrary apart from the singularities of A_a^w .

(ii) **Asymptotic decomposition.** Let $G(X, \Lambda)$, $\Psi(X, \Lambda)$ be from (2.10) and $\langle \mu \rangle$ from (2.5). Recall that $w_\Lambda(f(\Lambda)) = f(w^{-1}(\Lambda))$ and $w_\Lambda(\Lambda_b) = \Lambda_{w(b)}$ for $w \in W, b \in P$; also, $\Lambda^\iota = -w_0(\Lambda)$, $(\Lambda^\iota)_b = \Lambda_{b^\iota}$.

$$(2.16) \quad \text{Setting } \sigma_*(\Lambda) \stackrel{\text{def}}{=} \prod_{\alpha > 0} \prod_{j=1}^{\infty} \frac{1 - t_\alpha q_\alpha^j \Lambda_\alpha^{-1}}{1 - q_\alpha^j \Lambda_\alpha^{-1}}, \quad \Psi(X, \Lambda) = \\ = \langle \mu \rangle \sum_{w \in W} w_\Lambda \left(\sigma_*(\Lambda) \theta(\Lambda^\iota X q^{\rho_k}) \Xi^{(w)}(X, \Lambda) \right) \prod_{\alpha > 0 > w(\alpha)} \left(\frac{1 - t_\alpha \Lambda_{w(\alpha)}^{-1}}{1 - \Lambda_{w(\alpha)}^{-1}} \right),$$

provided the convergence of the right-hand side; $\sigma_*(\Lambda)$ is a modification of the q, t -generalization σ of the Harish-Chandra c -function from formula (4.5) in [Ch5] with $j \geq 1$ instead of $j \geq 0$ and $\Lambda \mapsto \Lambda^{-1}$.

An outline of the proof. The stabilization in (i) can be deduced directly from the eigenvalue problem (2.2), which is the definition of E_b in this paper. However it is more convenient to use the intertwining operators from [Ch3] (due to Knop and Sahi for A_n). One can also use the formulas from [HHL, RY, OS] based on the intertwining operators; see (the most general) Theorem 3.13 in [OS]. The intertwining operators are not creation operators any longer (as for the E -polynomials), but provide recurrence relations that are sufficient to justify (i). We will omit the details in this paper.

The key in (ii) is the verification that the asymptotic expansion of $\Psi(X, \Lambda)$ for $\Lambda = q^{b_\#} = q^{u_b^{-1}(b_- - \rho_k)}$ coincides with the term with $w = u_b^{-1}$ in the right-hand side of (2.16). Here $b = u_b^{-1}(b_-)$; see (2.2).

First of all, let us check that the multiplier

$$(2.17) \quad v(\sigma_*(\Lambda)) \prod_{\alpha > 0 > v(\alpha)} \left(\frac{1 - t_\alpha \Lambda_{v(\alpha)}^{-1}}{1 - \Lambda_{v(\alpha)}^{-1}} \right) \quad \text{for } v \in W$$

is nonzero only when $v = u_b^{-1}$ for such Λ . Indeed, if $v \neq u_b^{-1}$, there exists $\alpha > 0$ and a simple root α_i such that $u_b v(\alpha) = -\alpha_i$. This results in $(u_b^{-1}(\rho), v(\alpha)) = -1$ and $(u_b^{-1}(b_-), v(\alpha)) = (b_-, -\alpha_i) \geq 0$.

If $(b_-, \alpha_i) = 0$, then $u_b^{-1}(\alpha_i) > 0$ (we use the minimality of u_b) and $v(\alpha) < 0$, which is the inequality from the product part in (2.17). Therefore the term $(1 - q^j t_\alpha \Lambda_{v(\alpha)}^{-1})$ in the numerator of (2.17) will vanish at $\Lambda = q^{b_\#}$ for $j = -(b_-, \alpha_i) \geq 0$.

We will use the relation $\theta(Xq^b) = X_b^{-1} q^{-(b,b)/2} \theta(X)$ for $b \in P$, which follows from the P -periodicity of $q^{\frac{(x,x)}{2}} \theta(X)$ and readily results in

$$\theta(Xq^{b' - \rho_k} q^{\rho_k}) = q^{-(b, \rho_k)} X_{b'}^{-1} \frac{\theta(X) \theta(q^{b - \rho_k})}{\theta(q^{\rho_k})}.$$

Let us consecutively apply formulas (2.3), (2.12) and (2.5) (the Shintani formula, the evaluation and the constant term formulas). We will begin with $b = b_- \in P_-$, $w = \text{id}$; the case of general $b \in P$ is parallel.

We evaluate $\langle \mu \rangle \sigma_*(q^{b - \rho_k}) \theta(Xq^{b' - \rho_k} q^{\rho_k}) \Xi^{(\text{id})}(X, q^{b - \rho_k})$, which is the first and the only nonzero term in the decomposition from (2.16) for $\Lambda = q^{b_\#} = q^{b - \rho_k}$. Using (i) for $b = b_-$,

$$\begin{aligned} & \theta(Xq^{b' - \rho_k} q^{\rho_k}) \Xi^{(\text{id})}(X, q^{b - \rho_k}) = q^{-(b, \rho_k)} X_{b'}^{-1} (X_{b'} E_b) \frac{\theta(X) \theta(q^{b - \rho_k})}{\theta(q^{\rho_k})} \\ & = q^{-(b, \rho_k)} \left(E_b(q^{-\rho_k}) E_b' \right) \frac{\theta(X) \theta(q^{b - \rho_k})}{\theta(q^{\rho_k})} = q^{-(b, \rho_k)} E_b(q^{-\rho_k}) \\ & \quad \times \left(\prod_{\alpha > 0} \prod_{j=1}^{\infty} \left(\frac{1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha j}}{1 - q^{(\rho_k, \alpha) + \nu_\alpha j}} \right) G(X, q^{b - \rho_k}) \right) \frac{\theta(X) \theta(q^{b - \rho_k})}{\theta(q^{\rho_k})} \\ & = q^{-(b, \rho_k)} E_b(q^{-\rho_k}) \prod_{\alpha > 0} \prod_{j=1}^{\infty} \left(\frac{1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha j}}{1 - q^{(\rho_k, \alpha) + \nu_\alpha j}} \right) \Psi(X, q^{b - \rho_k}) \\ & = \prod_{\alpha > 0} \left(\prod_{j=1}^{-(b, \alpha^\vee)} \frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \prod_{j=1}^{\infty} \frac{1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha j}}{1 - q^{(\rho_k, \alpha) + \nu_\alpha j}} \right) \Psi(X, q^{b - \rho_k}). \end{aligned}$$

Thus,

$$\begin{aligned} & \langle \mu \rangle \sigma_*(q^{b - \rho_k}) \theta(Xq^{b' - \rho_k} q^{\rho_k}) \Xi^{(\text{id})}(X, q^{b - \rho_k}) / \Psi(X, q^{b - \rho_k}) \\ & = \prod_{\alpha > 0} \prod_{j=1}^{\infty} \frac{(1 - q_\alpha^j q^{(\rho_k, \alpha)})^2}{(1 - t_\alpha q_\alpha^j q^{(\rho_k, \alpha)}) (1 - t_\alpha^{-1} q_\alpha^j q^{(\rho_k, \alpha)})} \\ (2.18) \quad & \times \prod_{\alpha > 0} \prod_{j=1}^{\infty} \frac{1 - t_\alpha q_\alpha^j q^{-(b_-, \alpha) + (\rho_k, \alpha)}}{1 - q_\alpha^j q^{-(b_-, \alpha) + (\rho_k, \alpha)}} \\ & \times \prod_{\alpha > 0} \left(\prod_{j=1}^{-(b_-, \alpha^\vee)} \left(\frac{1 - t_\alpha q_\alpha^j q^{(\rho_k, \alpha)}}{1 - q_\alpha^j q^{(\rho_k, \alpha)}} \right) \prod_{j=1}^{\infty} \left(\frac{1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha j}}{1 - q^{(\rho_k, \alpha) + \nu_\alpha j}} \right) \right) = 1. \end{aligned}$$

This cancelation readily results from

$$\prod_{\alpha > 0} \left(\prod_{j=1}^{\infty} \frac{1 - t_{\alpha} q_{\alpha}^j q^{-(b_{-}, \alpha) + (\rho_k, \alpha)}}{1 - q_{\alpha}^j q^{-(b_{-}, \alpha) + (\rho_k, \alpha)}} \right)^{- (b_{-}, \alpha^{\vee})} \left(\frac{1 - t_{\alpha} q_{\alpha}^j q^{(\rho_k, \alpha)}}{1 - q_{\alpha}^j q^{(\rho_k, \alpha)}} \right) =$$

$$\prod_{\alpha > 0} \prod_{j=1}^{\infty} \left(\frac{1 - q_{\alpha}^j t_{\alpha} q^{(\rho_k, \alpha)}}{1 - q_{\alpha}^j q^{(\rho_k, \alpha)}} \right).$$

When $b \in P$ is arbitrary, let $w = u_b^{-1}$ and $\Lambda = u_b^{-1}(q^{-b_{-} - \rho_k})$. Then

$$\prod_{\alpha > 0 > w(\alpha)} \frac{1 - t_{\alpha} \Lambda_{w(\alpha)}^{-1}}{1 - \Lambda_{w(\alpha)}^{-1}} = \prod_{\alpha > 0 > w(\alpha)} \frac{1 - t_{\alpha} q^{-(b_{-}, \alpha) + (\rho_k, \alpha)}}{1 - q^{-(b_{-}, \alpha) + (\rho_k, \alpha)}}$$

must be added to (2.18). Respectively, the upper limit $-(b_{-}, \alpha^{\vee})$ for j (the last line there) must be diminished to $j(b, \alpha) = -(b_{-}, \alpha^{\vee}) - \xi(u_b^{-1}(\alpha))$, where $\xi(\alpha) = 0, 1$ for $\alpha \in R_{\pm}$; see (2.3). These adjustments complement each other and the result will be 1, as well as for $b = b_{-}$.

Next, we use the decomposition

$$(2.19) \quad \Psi(X, \Lambda) = \langle \mu \rangle \sum_{w \in W} w_{\Lambda} \left(\varpi_w(\Lambda) \theta(\Lambda^{\iota} X q^{\rho_k}) \Xi^{(w)}(X, \Lambda) \right),$$

where the coefficients ϖ_w do not depend on X . This is a general fact, which follows from the difference Dunkl eigenvalue problem in (2.11): $Y_a(G(X, \Lambda)) = \Lambda_a^{-1} G(X, \Lambda)$ for $a \in P$. Here you need to consider the latter in the W -spinors, treating the action of $w \in W$ as the corresponding permutation of the *independent* spinor components. Note that this is parallel to what was done in [Op] in the differential setting. The (formal) origin of the W -spinors are [Ch6] (where they were used for the Cherednik-Matsuo theorem) and [Op]; see [CM, CO2] for the exact definitions and a comprehensive discussion.

Here generally, ϖ_w can be P -periodic functions in terms of x . However in our setting, all functions are Laurent series in terms of X and therefore ϖ_w must be double periodic x -functions for $P \oplus \frac{2\pi i}{\log(q)} P$. Since they have no singularities in x for sufficiently large negative $\Re(x)$ due to the convergence of Ψ , they do not depend of X .

We conclude that for the Kronecker delta $\delta_{v,w}$,

$$(2.20) \quad w(\varpi_v(\Lambda)) = \delta_{v,w} w(\sigma_*(\Lambda)) \prod_{\alpha > 0 > w(\alpha)} \left(\frac{1 - t_{\alpha} \Lambda_{w(\alpha)}^{-1}}{1 - \Lambda_{w(\alpha)}^{-1}} \right), \text{ where}$$

$$\Lambda = w(q^{b_{-} - \rho_k}) = q^{b_{\#}} \text{ for } b \in P, \quad w = u_b^{-1}, \quad b_{-} = u_b(b) \in P_{-}.$$

Relations from (2.20) are actually sufficient to fix $\{\varpi_w\}$ uniquely via the analyticity considerations. A usual (classical) justification of similar facts is what we did for A_1 , using the $X \leftrightarrow \Lambda$ -symmetry and considering the asymptotic sectors where exactly one of the terms in the decomposition (2.16) survives. One needs to know in this approach the limits of $X_{w(b_-)}E_{b_-}$ as $|X_{w(\alpha)}| \rightarrow \infty$ for all $\alpha > 0$; cf. (1.20). These limits are relatively straightforward to find. As in [Ch5], we obtain the recurrence relations for them (coinciding with those for the corresponding ϖ_w). We will omit the details in this paper. \square

An orbit-sum formula. There is an interesting application of the decomposition theorem to the E -polynomials, which is another (“global”) relation connecting E_b and $\Xi^{(w)}$. Using the $X \leftrightarrow \Lambda$ -symmetry of $G(X, \Lambda)$, we can substitute $X = q^{b_- - \rho_k}$ in (2.16). One has:

$$\begin{aligned}
 (2.21) \quad & \theta(\Lambda)^{-1} q^{\frac{(b,b)}{2}} \Psi(q^{b_- - \rho_k}, \Lambda) \\
 &= \prod_{\alpha > 0} \left(\prod_{j=1}^{-(b_-, \alpha^\vee)} \left(\frac{1 - t_\alpha q_\alpha^j q^{(\rho_k, \alpha)}}{1 - q_\alpha^j q^{(\rho_k, \alpha)}} \right) \prod_{j=1}^{\infty} \left(\frac{1 - t_\alpha^{-1} q_\alpha^j q^{(\rho_k, \alpha)}}{1 - q_\alpha^j q^{(\rho_k, \alpha)}} \right) \right)^{-1} E_b(\Lambda) \\
 &= \langle \mu \rangle \sum_{w \in W} w_\Lambda \left(\sigma_*(\Lambda) \frac{\theta(\Lambda^\vee q^{b_- - \rho_k} q^{\rho_k})}{\theta(\Lambda) q^{-\frac{(b,b)}{2}}} \Xi^{(w)}(q^{b_- - \rho_k}, \Lambda) \right) \prod_{\alpha > 0 > w(\alpha)} \frac{1 - t_\alpha \Lambda_{w(\alpha)}^{-1}}{1 - \Lambda_{w(\alpha)}^{-1}} \\
 &= \langle \mu \rangle \sum_{w \in W} w_\Lambda \left(\sigma_*(\Lambda) \Lambda_{b_-}^{-1} \Xi^{(w)}(q^{b_- - \rho_k}, \Lambda) \right) \prod_{\alpha > 0 > w(\alpha)} \frac{1 - t_\alpha \Lambda_{w(\alpha)}^{-1}}{1 - \Lambda_{w(\alpha)}^{-1}}.
 \end{aligned}$$

Replacing $\langle \mu \rangle$ by the corresponding product, we obtain the following identity:

$$\begin{aligned}
 (2.22) \quad & \prod_{\alpha > 0} \prod_{j=1}^{\infty} \left(\frac{1 - t_\alpha q_\alpha^j q^{(\rho_k - b_-, \alpha)}}{1 - q_\alpha^j q^{(\rho_k - b_-, \alpha)}} \right) E_{b_-}(\Lambda) = \sigma_*(q^{b_- - \rho_k}) E_{b_-}(\Lambda) \\
 &= \sum_{w \in W} w_\Lambda \left(\sigma_*(\Lambda) \Lambda_{b_-}^{-1} \Xi^{(w)}(q^{b_- - \rho_k}, \Lambda) \right) \prod_{\alpha > 0 > w(\alpha)} \frac{1 - t_\alpha \Lambda_{w(\alpha)}^{-1}}{1 - \Lambda_{w(\alpha)}^{-1}}.
 \end{aligned}$$

It is instructional to plug in here $\Lambda = q^{b_\#}$ for $b = u_b^{-1}(b_-)$. As we know, only the term with $w = u_b^{-1}$ in the last sum will not vanish and $\sigma_*(q^{b_- - \rho_k}) = \sigma_*(q^{u_b(b_\#)}) = u_b^{-1}(\sigma_*(q^{b_\#}))$; recall that $b_\# = u_b^{-1}(b_- - \rho_k)$. Then $u_b^{-1}(E_b(q^{b_\#})) = E_b(q^{b_- - \rho_k})$, $\Lambda_{w(\alpha)}^{-1} = q^{(\rho_k - b_-, \alpha)}$ and we arrive at

$$E_{b_-}(q^{b_\#}) = \prod_{\alpha > 0 > u_b^{-1}(\alpha)} \frac{1 - t_\alpha q^{(\rho_k - b_-, \alpha)}}{1 - q^{(\rho_k - b_-, \alpha)}} E_b(q^{b_- - \rho_k}).$$

This is a particular case of the Duality Theorem from [Ch2]:

$$E_b(q^{c\sharp})E_c(q^{-\rho_k}) = E_c(q^{b\sharp})E_b(q^{-\rho_k}), \quad b, c \in P.$$

Indeed,

$$E_{b_-}(q^{-\rho_k})/E_b(q^{-\rho_k}) = \prod_{\alpha > 0 > u_b^{-1}(\alpha)} \frac{1 - t_\alpha q^{(\rho_k - b_-, \alpha)}}{1 - q^{(\rho_k - b_-, \alpha)}},$$

which is direct from (2.3).

Note that (2.22) becomes very explicit for A_1 , which we will omit. Generally, the relation to the case of A_1 is as follows:

$$\Xi^{(\text{id})} = \tilde{\Xi}_-, \quad \Xi^{(s)} = \tilde{\Xi}_+; \quad \text{see (1.16)}.$$

Let us provide two examples of the coefficients of Ξ^w beyond A_1 . We use the formula from [HHL] with some help of *SAGE*. For A_2 , let us calculate the constant term of $\Xi^{(\text{id})}$, which is the coefficient of X_{b_+} in the polynomial E_{b_-} ($b_- \in P_-$) upon the substitution $\Lambda = q^{b_- - \rho_k}$ for any $b_- \in P_-$. It equals

$$\Xi^{(\text{id})}(X_\alpha \rightarrow \infty, \alpha > 0) = \frac{(1-t)(1-t+t^2-t(\Lambda_{\alpha_1}^{-1} + \Lambda_{\alpha_2}^{-1} - \Lambda_{\alpha_1+\alpha_2}^{-1}))}{(1-\Lambda_{\alpha_1}^{-1})(1-\Lambda_{\alpha_2}^{-1})(1-\Lambda_{\alpha_1+\alpha_2}^{-1})}.$$

The coefficient of $\Xi^{(\text{id})}$ of $X_{\alpha_1}^{-1}$ equals

$$\frac{q(1-t)^2(1-t\Lambda_{\alpha_2})(1-t+qt^2-t(q\Lambda_{\alpha_1}^{-1} + \Lambda_{\alpha_2}^{-1} - \Lambda_{\alpha_1+\alpha_2}^{-1}))}{t(1-q)(1-q\Lambda_{\alpha_2})(1-\Lambda_{\alpha_1}^{-1})(1-\Lambda_{\alpha_2}^{-1})(1-\Lambda_{\alpha_1+\alpha_2}^{-1})}.$$

Accordingly, the coefficient of $\Xi^{(\text{id})}$ of $X_{\alpha_2}^{-1}$ is obtained by the transposition $\alpha_1 \leftrightarrow \alpha_2$ in the formula above.

Symmetrization. The passage from the decomposition (2.16) to that of $F(X, \Lambda)$ can be now performed by means of the connection formula (2.14). Then we will arrive at Theorem 4.6 from [St2] (in the reduced case). We do not provide here the formula for the radius of convergence of $\Xi^{(w)}(X, \Lambda)$ in terms of X . Similar to [St2], this can be generally achieved using the $X \leftrightarrow \Lambda$ -duality and calculating the distance to the first Λ -singular point in the σ -decomposition.

The σ -decomposition of $F(X, \Lambda)$ in the case of $t = 0$ is directly related to [GL, BF] (the q -Whittaker case) and the so-called K -theoretic J -function of flag varieties. For arbitrary t , the connection with the Laumon spaces was established for A_n in [BFS]; see Conjecture 1.8 and especially Proposition 5.11 there.

No geometric interpretation of the E -polynomials is known for general parameters at the moment. It does exist in the following cases.

First of all, E -polynomials coincide with the p -adic spherical (non-symmetric) Matsumoto functions for $q = 0$. Then there is a direct link to the level-one Demazure characters [San, Ion] in the twisted case for $t = 0$. More recently, the global *nonsymmetric* q -Whittaker function and the E -polynomials at $t = \infty$ appeared connected with so-called PBW-filtration; see [CO2, CF].

Generally, we have a powerful nonsymmetric machinery of intertwining operators [Ch3, HHL, RY, OS], which does not exist in the symmetric theory, but an obvious lack of geometric understanding of the E -polynomials (so far).

REFERENCES

- [BF] A. Braverman, and M. Finkelberg, *Semi-infinite Schubert varieties and quantum K-theory of flag manifolds*, arXiv:1111.2266 (2011). [18](#)
- [BFS] — , and — , and J. Shiraishi, *Macdonald polynomials, Laumon spaces and perverse coherent sheaves*, arxiv:1206.3131v2 (2013). [3](#), [18](#)
- [Ch1] I. Cherednik, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series, 319, Cambridge University Press, Cambridge, 2006. [3](#), [9](#), [10](#), [11](#), [12](#), [13](#)
- [Ch2] — , *Nonsymmetric Macdonald polynomials* , IMRN 10 (1995), 483–515. [4](#), [10](#), [11](#), [12](#), [18](#)
- [Ch3] — , *Intertwining operators of double affine Hecke algebras*, Selecta Math. New ser. 3 (1997), 459–495. [2](#), [4](#), [14](#), [19](#)
- [Ch4] — , *Difference Macdonald-Mehta conjecture*, IMRN 10 (1997), 449–467. [2](#), [5](#), [6](#), [7](#), [10](#), [12](#)
- [Ch5] — , *Whittaker limits of difference spherical functions*, IMRN, 20 (2009), 3793–3842; arXiv:0807.2155 (2008). [2](#), [3](#), [4](#), [6](#), [9](#), [10](#), [14](#), [17](#)
- [Ch6] — , *Integration of quantum many-body problems by affine Knizhnik–Zamolodchikov equations*, preprint RIMS 776 (1991), Adv. Math. 106 (1994), 65–95. [16](#)
- [CM] — , and X. Ma, *Spherical and Whittaker functions via DAHA II (Spinor Whittaker and Bessel functions in rank one)*, Selecta Mathematica (2012); Preprint arXiv: 0904.4324 (2009). [2](#), [3](#), [5](#), [16](#)
- [CO1] — , and D. Orr, *One-dimensional nil-DAHA and Whittaker functions I*, Transformation Groups 17:4 (2012), 953–987. [2](#), [3](#), [4](#), [5](#), [7](#)
- [CO2] — , and — , *Nonsymmetric difference Whittaker functions*, arxiv: 1302.4094 [math.QA] (2013). [2](#), [16](#), [19](#)
- [CF] — , and E. Feigin, *Extremal part of the PBW-filtration and E-polynomials*, arXiv:1306.3146 [math.QA] (2013). [19](#)

- [GL] A. Givental, and Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. Math., 151 (2003), 193–219. [18](#)
- [HHL] J. Haglund, and M. Haiman, and N. Loehr, *A combinatorial formula for nonsymmetric Macdonald polynomials*, Amer. J. Math. 130:2 (2008), 359–383. [2](#), [14](#), [18](#), [19](#)
- [HO] G.J. Heckman, and E.M. Opdam, *Root systems and hypergeometric functions I*, Comp. Math. 64 (1987), 329–352. [2](#)
- [Ion] B. Ion, *Nonsymmetric Macdonald polynomials and Demazure characters*, Duke Mathematical Journal 116:2 (2003), 299–318. [19](#)
- [Ma] I. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Séminaire Bourbaki 47:797 (1995), 1–18. [4](#), [12](#)
- [MS] M. van Meer, and J.V. Stokman, *Double affine Hecke algebras and bispectral quantum Knizhnik- Zamolodchikov equations*, Int. Math. Res. Not., 6, (2010) 969–1040. [3](#)
- [Op] E. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. 175 (1995), 75–121. [2](#), [4](#), [16](#)
- [OS] D. Orr, and M. Shimozono, *Specializations Of nonsymmetric Macdonald-Koornwinder polynomials*, arxiv.org/abs/1310.0279v2 (2013). [2](#), [14](#), [19](#)
- [RY] A. Ram, and M. Yip, *A combinatorial formula for Macdonald polynomials*, Adv. Math. 226:1 (2011), 309–331. [2](#), [14](#), [19](#)
- [San] Y. Sanderson, *On the Connection Between Macdonald Polynomials and Demazure Characters*, J. of Algebraic Combinatorics, 11 (2000), 269–275. [19](#)
- [St1] J. Stokman, *Difference Fourier transforms for nonreduced root systems*, Sel. math., New ser. 9 (2003) 409–494. [10](#)
- [St2] J. Stokman *The c-function expansion of a basic hypergeometric function associated to root systems*, Annals of Math. 179:1, 253–299. [2](#), [3](#), [18](#)

(I. Cherednik) DEPARTMENT OF MATHEMATICS, UNC CHAPEL HILL, NORTH CAROLINA 27599, USA, CHERED@EMAIL.UNC.EDU